Generalized Gorenstein Arf rings

Naoki Taniguchi

Meiji University

Based on the works jointly with

Ela Celikbas, Olgur Celikbas, and Shiro Goto

September 5, 2017

Introduction

Aim of this research

Find a new class of Cohen–Macaulay rings, which naturally covers the class of Gorenstein rings and fills in a gap between Cohen–Macaulay and Gorenstein properties.

History of almost Gorenstein rings

- [Barucci-Fröberg, 1997]

  $\cdots$ one-dimensional analytically unramified local rings

- [Goto-Matsuoka-Phuong, 2013]

  $\cdots$ one-dimensional Cohen–Macaulay local rings

- [Goto-Takahashi-T, 2015]

  $\cdots$ higher-dimensional Cohen–Macaulay local/graded rings

Recently, S. Goto and S. Kumashiro defined

Generalized Gorenstein local rings

Theorem 1.1 (Barucci-Fröberg, 1997)

Let $0 < a_1 < a_2 < \cdots < a_\ell \in \mathbb{Z}$ $(\ell > 0)$ *s.t.* $\gcd(a_1, a_2, \ldots, a_\ell) = 1$. *Let* $V = k[[t]]$ *be the formal power series ring over a field* k. *We set*

$$R = k[[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}]] \ \ \text{and} \ \ H = \langle a_1, a_2, \ldots, a_\ell \rangle .$$

Then TFAE.

(1) *R is an almost Gorenstein Arf ring.*

(2) $2 + a_i \in H$ *for* $1 \leq \forall i \leq \ell$.

Question 1.2

Is there any characterization of generalized Gorenstein Arf rings?

Theorem 1.1 (Barucci-Fröberg, 1997)

Let $0 < a_1 < a_2 < \cdots < a_\ell \in \mathbb{Z}$ $(\ell > 0)$ *s.t.* $\gcd(a_1, a_2, \ldots, a_\ell) = 1$. *Let* $V = k[[t]]$ *be the formal power series ring over a field* k. *We set*

$$R = k[[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}]] \text{ and } H = \langle a_1, a_2, \ldots, a_\ell \rangle .$$

Then TFAE.

(1) R *is an almost Gorenstein Arf ring.*

(2) $2 + a_i \in H$ *for* $1 \leq \forall i \leq \ell$.

Question 1.2

Is there any characterization of generalized Gorenstein Arf rings?

Setting 1.3

- $(R, \mathfrak{m})$ a Cohen–Macaulay local ring with $d = \dim R$
- $|R/\mathfrak{m}| = \infty$
- $\exists$ K_R the canonical module of R

Definition 1.4 (Goto-Kumashiro, 2017)

We say that R is *a generalized Gorenstein ring*, if either (1) R is a Gorenstein ring, or (2) R is not a Gorenstein ring, but $\exists$ $\sqrt{\mathfrak{a}} = \mathfrak{m}$, $\exists$ an exact sequence

$$0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$$

of R-modules s.t. C is an Ulrich R-module w.r.t. $\mathfrak{a}$, and

$$R/\mathfrak{a} \otimes_R \varphi : R/\mathfrak{a} \to \mathrm{K}_R/\mathfrak{a}\,\mathrm{K}_R$$

is injective.

Let M be a finite R-module with $s = \dim_R M \geq 0$, $\sqrt{\mathfrak{a}} = \mathfrak{m}$.

Definition 1.5 (Goto-Ozeki-Takahashi-Watanabe-Yoshida, 2014)

We say that M is *an Ulrich R-module with respect to $\mathfrak{a}$* if

- M is a CM R-module,
- $e^0_{\mathfrak{a}}(M) = \ell_R(M/\mathfrak{a}M)$, and
- $M/\mathfrak{a}M$ is a free $R/\mathfrak{a}$-module.

Note that if M is a CM R-module, then

$$e^0_{\mathfrak{a}}(M) = \ell_R(M/\mathfrak{a}M) \iff \mathfrak{a}M = (f_1, f_2, \dots, f_s)M \text{ for } \exists\, f_1, f_2, \dots, f_s \in \mathfrak{a}$$

Suppose that $\exists$ an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules s.t. $C \neq (0)$. Then C is a CM R-module with $\dim_R C = d - 1$.

Note that C is an Ulrich R-module w.r.t $\mathfrak{a}$ if and only if

- $C/\mathfrak{a}C$ is a free $R/\mathfrak{a}$-module and
- $\mathfrak{a}C = (f_2, f_3, \ldots, f_d)C$ for $\exists$ $f_2, f_3, \ldots, f_d \in \mathfrak{a}$.

Therefore, if $\mathfrak{a} = \mathfrak{m}$,

an almost Gorenstein ring $\Longrightarrow$ a generalized Gorenstein ring

Example 1.6 (Goto-Kumashiro, 2017)

- R is a generalized Gorenstein ring, if $\mathrm{e}(R) \leq 3$.
- Let $R = k[[X, Y, Z, W]]/\mathrm{I}_2 \begin{pmatrix} X^n & Y & Z \\ Y & Z & W \end{pmatrix}$ $(n > 0)$. Then R is a generalized Gorenstein ring w.r.t. $\mathfrak{a}_i = (x^i, y, z, w)$ for $1 \leq i \leq n$.

Setting 1.7

- $(R, \mathfrak{m})$ a Cohen–Macaulay local ring with $\dim R = 1$
- $\exists$ K_R the canonical module of R
- I a canonical ideal of R, that is, $I \neq R$ and $I \cong \mathrm{K}_R$
- $Q = (a)$ a minimal reduction of I
- $K = \frac{I}{a} = \left\{ \frac{x}{a} \;\middle|\; x \in I \right\} \subseteq \mathrm{Q}(R)$
- $S = R[K]$
- $\mathfrak{c} = R : S$

Then, K is a fractional ideal of R s.t.

$$R \subseteq K \subseteq \overline{R}$$

and S is a module-finite extension of R.

The main result of this talk is stated as follows.

Theorem 1.8 (Celikbas-Celikbas-Goto-T, 2017)

Suppose that R is a generalized Gorenstein ring. Then TFAE.

(1) *R is an Arf ring.*

(2) $v(R) = \mathrm{e}(R)$ *and* $\mathrm{e}(S_M) \le 2$ *for* $\forall M \in \operatorname{Max} S$.

Survey on Arf rings

Let A be a commutative Noetherian semi-local ring and assume that

($\sharp$) *A_M is a CM local ring with* $\dim A_M = 1$ *for* $\forall M \in \operatorname{Max} A$.

Let $\mathcal{F}_A = \{I \subseteq A \mid \exists \text{ NZD} \in I\}$. Then

$$A \subseteq I : I \subseteq I^2 : I^2 \subseteq \cdots \subseteq I^n : I^n \subseteq \cdots \subseteq \overline{A}$$

for $\forall I \in \mathcal{F}_A$. We set

$$A^I = \bigcup_{n>0} [I^n : I^n].$$

Note that if $x \in I$ is a reduction of I, then

$$A^I = A\left[\frac{I}{x}\right] \subseteq \mathrm{Q}(A).$$

We say that $I \in \mathcal{F}_A$ is *a stable ideal of A*, if $A^I = I : I$. Then

$$I \in \mathcal{F}_A \text{ is stable} \iff I^2 = xI \text{ for } \exists x \in I.$$

Definition 2.1 (Lipman, 1971)

The ring A is called *an Arf ring*, if every integrally closed ideal $I \in \mathcal{F}_A$ is stable.

Example 2.2

Let k be a field and $V = k[[t]]$ the formal power series ring.

- $R = k[[t^4, t^7, t^9, t^{10}]]$ is an Arf ring.
- $R = k[[t^3, t^7, t^{11}]]$ is NOT an Arf ring.

Notation 2.3

For $n \geq 0$, we set

$$A_n = \begin{cases} A & \text{if } n = 0, \\ A_{n-1}^{J(A_{n-1})} & \text{if } n > 0, \end{cases}$$

where $J(R)$ denotes the Jacobson radical of a ring R.

Theorem 2.4 (Lipman, 1971)

TFAE.

(1) *A is an Arf ring.*

(2) *For $\forall n \geq 0$, $\forall M \in \operatorname{Max} A_n$, we have $v((A_n)_M) = \mathrm{e}((A_n)_M)$.*

Proof of Theorem 1.8

We maintain the notation as in Setting 1.7 and Notation 2.3.

Setting 1.7

- $(R, \mathfrak{m})$ *a Cohen–Macaulay local ring with* $\dim R = 1$
- $\exists$ K_R *the canonical module of* R
- I *a canonical ideal of* R*, that is,* $I \neq R$ *and* $I \cong \mathrm{K}_R$
- $Q = (a)$ *a minimal reduction of* I
- $K = \frac{I}{a} = \left\{ \frac{x}{a} \;\middle|\; x \in I \right\} \subseteq \mathrm{Q}(R)$
- $S = R[K]$
- $\mathfrak{c} = R : S$

Theorem 3.1 (Goto-Matsuoka-Phuong, 2013)

TFAE.

(1) *R is an almost Gorenstein ring and $v(R) = \mathrm{e}(R)$.*

(2) *$B = \mathfrak{m} : \mathfrak{m}$ is a Gorenstein ring.*

Theorem 3.2 (Goto-Kumashiro, 2017)

Suppose that $\exists\ x \in \mathfrak{m}$ s.t. $\mathfrak{m}^2 = x\mathfrak{m}$. Then TFAE.

(1) *R is a generalized Gorenstein ring, but not an almost Gorenstein ring.*

(2) *$B = \mathfrak{m} : \mathfrak{m}$ is a non-Gorenstein generalized Gorenstein ring and $\mathfrak{n}^2 = x\mathfrak{n}$.*

When this is the case, $R/\mathfrak{m} \cong B/\mathfrak{n}$ and

$$\ell_B(B/(B : B[L])) = \ell_R(R/\mathfrak{c}) - 1$$

where $\mathfrak{n} = J(B)$ and $L = KB$.

Theorem 3.3

Suppose that $\mathrm{e}(R) \geq 3$. *Then TFAE.*

(1) R *is a generalized Gorenstein ring and* $v(R) = \mathrm{e}(R)$.

(2) S *is a Gorenstein ring and* $\exists N > 0$ *s.t.*

  (i) $R_N = S$, *and*

  (ii) R_n *is a local ring,* $v(R_n) = \mathrm{e}(R_n) = \mathrm{e}(R)$ *for* $0 \leq \forall n < N$.

When this is the case, we have $N = \ell_R(R/\mathfrak{c})$.

- R is a Gorenstein ring $\iff$ $\ell_R(R/\mathfrak{c}) = 0$
- R is a non-Gorenstein almost Gorenstein ring $\iff$ $\ell_R(R/\mathfrak{c}) = 1$

We set $\ell = \ell_R(R/\mathfrak{c})$, $e = \mathrm{e}(R)$, and $B = \mathfrak{m} : \mathfrak{m}$.

Proof of Theorem 3.3 (1) $\Rightarrow$ (2).

Choose $\exists\ x \in \mathfrak{m}$ s.t. $\mathfrak{m}^2 = x\mathfrak{m}$. Then $\ell > 0$, since R is not Gorenstein. If $\ell = 1$, then R is an almost Gorenstein ring and $S = B = R_1$ is a Gorenstein ring.

Suppose $\ell > 1$ and the assertion holds for $\ell - 1$. Since R is not an almost Gorenstein ring, B is a non-Gorenstein generalized Gorenstein ring and

$$\mathfrak{n}^2 = x\mathfrak{n} \quad \text{and} \quad R/\mathfrak{m} \cong B/\mathfrak{n}$$

where $\mathfrak{n} = J(B)$. Hence $\mathrm{e}^0_{\mathfrak{n}}(B) = e$.

Note that $B \subseteq L := KB \subseteq \overline{B}$, $L \cong \mathrm{K}_B$, and $S = B[L]$. Therefore

$$\ell_B(B/(B : B[L])) = \ell_R(R/\mathfrak{c}) - 1 = \ell - 1.$$

Hence $S = B[L]$ is a Gorenstein ring, $R_n = B_{n-1}$ is local, and $v(R_n) = \mathrm{e}(R_n) = e$ for $1 \leq \forall n < \ell$. $\square$

Proof of Theorem 3.3 (2) $\Rightarrow$ (1).

If $N = 1$, then $S = \mathfrak{m} : \mathfrak{m}$, so that R is an almost Gorenstein ring and $\ell = N$.

Suppose $N > 1$ and the assertion holds for $N - 1$. Since $v(R) = e$, $\exists\ x \in \mathfrak{m}$ s.t. $\mathfrak{m}^2 = x\mathfrak{m}$. Then $R_1 = B$ is local, $v(B) = \mathrm{e}(B) = e$, and

$$e = \mathrm{e}^0_{\mathfrak{m}}(R) = \mathrm{e}^0_{\mathfrak{m}}(B) = \ell_R(B/xB) \geq \ell_B(B/xB) \geq \mathrm{e}^0_{\mathfrak{n}}(B) = e$$

where $\mathfrak{n} = J(B)$. Hence $R/\mathfrak{m} \cong B/\mathfrak{n}$. Therefore $\mathfrak{n}^2 = x\mathfrak{n}$.

Since B is a non-Gorenstein generalized Gorenstein ring, R is a generalized Gorenstein ring, but not an almost Gorenstein ring, and

$$\ell_B(B/(B : B[KB])) = \ell - 1.$$

Since $v(R_n) = \mathrm{e}(R_n) = e$ for $1 \leq \forall n < N$, so is B_{n-1}. By induction hypothesis, $N - 1 = \ell - 1$, as desired. $\square$

Let us note the following.

Lemma 3.4

Let $R \subseteq C \subseteq \mathrm{Q}(R)$ be an intermediate ring s.t. C is a finite R-module. Suppose that $\mathrm{e}(C_M) \leq 2$ for $\forall M \in \operatorname{Max} C$. Then C is an Arf ring.

We are now ready to prove Theorem 1.8.

Theorem 1.8

Suppose that R is a generalized Gorenstein ring. Then TFAE.

(1) *R is an Arf ring.*

(2) $v(R) = \mathrm{e}(R)$ *and* $\mathrm{e}(S_M) \leq 2$ *for* $\forall M \in \operatorname{Max} S$.

Proof of Theorem 1.8.

May assume that $v(R) = \mathrm{e}(R)$ and $\mathrm{e}(R) \geq 3$. By Theorem 3.3, S is Gorenstein, $\exists N > 0$ s.t. $R_N = S$, R_n is local, and $v(R_n) = \mathrm{e}(R_n) = \mathrm{e}(R)$ for $0 \leq \forall n < N$.

Suppose that R is Arf. Since $S = R_N$, S is an Arf ring. Thus $\mathrm{e}(S_M) \leq 2$ for $\forall M \in \operatorname{Max} S$.

Conversely, we assume $\mathrm{e}(S_M) \leq 2$ for $\forall M \in \operatorname{Max} S$. By Lemma 3.4, S is Arf. Since R_n is local, $v(R_n) = \mathrm{e}(R_n) = \mathrm{e}(R)$ for $0 \leq \forall n < N$, R is Arf. □

We set $B = \mathfrak{m} : \mathfrak{m}$.

Corollary 3.5

TFAE.

(1) *R is an almost Gorenstein Arf ring.*

(2) $\mathrm{e}(B_M) \leq 2$ *for* $\forall M \in \operatorname{Max} B$.

Example 3.6

Let k be a field and $V = k[[t]]$.

- $R = k[[t^4, t^7, t^9, t^{10}]]$ is an Arf ring, but not generalized Gorenstein.
- $R = k[[t^3, t^7, t^{11}]]$ is an almost Gorenstein ring but not Arf.

We now recover the result of V. Barucci and R. Fröberg.

Theorem 1.1 (Barucci-Fröberg, 1997)

Let $R = k[[t^{a_1}, t^{a_2}, \dots, t^{a_\ell}]] \subseteq k[[t]]$ *and* $H = \langle a_1, a_2, \dots, a_\ell \rangle$. *Then TFAE.*

(1) R *is an almost Gorenstein Arf ring.*

(2) $2 + a_i \in H$ *for* $1 \le \forall i \le \ell$.

Corollary 3.7

Suppose that $R = k[[t^{a_1}, t^{a_2}, \dots, t^{a_\ell}]]$ *is a generalized Gorenstein ring. Then* R *is an Arf ring if and only if*

- $v(R) = \mathrm{e}(R)$,
- $2 + \ell_R(R/\mathfrak{c}) \cdot a_1 \in H$, *and*
- $2 + a_i \in H$ *for* $2 \le \forall i \le \ell$.

Example 3.8

Let k be a field and $V = k[[t]]$.

(1) $R = k[[t^{e+i} \mid 0 \leq i \leq e-1]]$ $(e \geq 2)$ is an almost Gorenstein Arf ring.

(2) $R = k[[t^e, \{t^{e+i} \mid 2 \leq i \leq e-1\}, t^{2e+1}]]$ $(e \geq 3)$ is an almost Gorenstein Arf ring.

(3) $R = k[[t^5, t^{16}, t^{17}, t^{18}, t^{19}]]$ is a generalized Gorenstein Arf ring, which is not an almost Gorenstein ring.

Application

Let $A = R \ltimes \mathfrak{c}$ be the idealization of $\mathfrak{c} = R : S$.

Theorem 4.1

Suppose that R is a generalized Gorenstein ring. Then TFAE.

(1) *A is an Arf ring.*

(2) *$\mathrm{v}(R) = \mathrm{e}(R)$ and $S = \overline{R}$.*

When this is the case, R is an Arf ring.

As a consequence, we have the following.

Corollary 4.2

$R \ltimes \mathfrak{m}$ is an almost Gorenstein Arf ring if and only if $\mathfrak{m}\overline{R} \subseteq R$.

Thank you so much for your attention.